Stanley depth of the path ideal associated to a line graph

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Abstract

We consider the path ideal associated to a line graph, we compute `sdepth` for its quotient ring and note that it is equal with its `depth`. In particular, it satisfies the Stanley inequality.

Keywords: Stanley depth, Stanley inequality, path ideal, line graph, simplicial tree.
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Introduction

Let K be a field and $S = K[x_1, \ldots, x_n]$ the polynomial ring over K. Let M be a $\mathbb{Z}^n$-graded S-module. A *Stanley decomposition* of M is a direct sum $\mathcal{D} : M = \bigoplus_{i=1}^r m_i K[Z_i]$ as a $\mathbb{Z}^n$-graded K-vector space, where $m_i \in M$ is homogeneous with respect to $\mathbb{Z}^n$-grading, $Z_i \subset \{x_1, \ldots, x_n\}$ such that $m_i K[Z_i] = \{um_i : u \in K[Z_i]\} \subset M$ is a free $K[Z_i]$-submodule of M. We define $\operatorname{sdepth}(\mathcal{D}) = \min_{i=1,\ldots,r} |Z_i|$ and $\operatorname{sdepth}_S(M) = \max\{\operatorname{sdepth}(\mathcal{D}) |\ \mathcal{D}$ is a Stanley decomposition of $M\}$. The number $\operatorname{sdepth}_S(M)$ is called the *Stanley depth* of M. In [1], J. Apel restated a conjecture firstly given by Stanley in [16], namely that $\operatorname{sdepth}_S(M) \geq \operatorname{depth}_S(M)$ for any $\mathbb{Z}^n$-graded S-module M. This conjecture proves to be false, in general, for $M = S/I$ and $M = J/I$, where $0 \neq I \subset J \subset S$ are monomial ideals, see [7].

Herzog, Vladoiu and Zheng show in [11] that $\operatorname{sdepth}_S(M)$ can be computed in a finite number of steps if $M = I/J$, where $J \subset I \subset S$ are monomial ideals. In [15], Rinaldo give a computer implementation for this algorithm, in the computer algebra system *CoCoA* [6]. However, it is difficult to compute this invariant, even in some very particular cases. For instance in [2] Biro et al. proved that $\operatorname{sdepth}(\mathbf{m}) = \lceil n/2 \rceil$ where $\mathbf{m} = (x_1, \ldots, x_n)$. For a friendly introduction on Stanley depth we recommend [12].

Let $\Delta \subset 2^{[n]}$ be a simplicial complex. A face $F \in \Delta$ is called a *facet*, if F is maximal with respect to inclusion. We denote $\mathcal{F}(\Delta)$ the set of facets of Δ. If $F \in \mathcal{F}(\Delta)$, we denote $x_F = \prod_{j\in F} x_j$. Then the *facet ideal* $I(\Delta)$ associated to Δ is the squarefree monomial ideal $I = (x_F\ :\ F \in \mathcal{F}(\Delta))$ of S. The facet ideal was studied by Faridi [8] from the `depth` perspective.

A line graph of lenght n, denoted by L_n, is a graph with the vertex set $V = [n]$ and the edge set $E = \{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}\}$. The Stanley depth of the edge ideal associated to L_n (which is in fact the facet ideal of L_n, if we look at L_n as a simplicial complex) was computed by Alin Ştefan in [17].

Let $\Delta_{n,m}$ be the simplicial complex with the set of facets $\mathcal{F}(\Delta_{n,m}) = \{\{1, 2, \ldots, m\}, , \{2, 3, \ldots, m + 1\}, \cdots , \{n - m + 1, n - m + 2, \ldots, n\}\}$. We denote $I_{n,m} = (x_1x_2 \cdots x_m, x_2x_3 \cdots x_{m+1}, \ldots, x_{n-m+1}x_{n-m+2} \cdots x_n)$, the associated facet ideal.

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Note that $I_{n,m}$ is the path ideal of the graph L_n, provided with the direction given by $1 < 2 < \ldots < n$, see [10] for further details.

According to [10, Theorem 1.2],

$$pd(S/I_{n,m}) = \begin{cases} \frac{2(n-d)}{m+1}, \ n \equiv d(mod\ (m+1))\ with\ 0 \leq d \leq m-1, \\ \frac{2n-m+1}{m+1}, \ n \equiv m(mod\ (m+1)). \end{cases}$$

By Auslander-Buchsbaum formula (see [19]), it follows that $\operatorname{depth}(S/I_{n,m}) = n - pd(S/I_{n,m})$ and, by a straightforward computation, we can see $\operatorname{depth}(S/I_{n,m}) = n+1-\left\lfloor \frac{n+1}{m+1} \right\rfloor - \left\lceil \frac{n+1}{m+1} \right\rceil$.

We prove that $\operatorname{sdepth}(S/I_{n,m}) = \operatorname{depth}(S/I_{n,m}) = n+1-\left\lfloor \frac{n+1}{m+1} \right\rfloor - \left\lceil \frac{n+1}{m+1} \right\rceil$, see Theorem 1.3. In particular, we give another prove for the result of [10, Theorem 1.2]. Also, our result generalize [17, Lemma 4].

We recall some notions introduced by Faridi in [8]. Let Δ be a simplicial complex. A facet F of Δ is called a *leaf*, if either F is the only facet of Δ, or there exists a facet G in Δ, $G \neq F$, such that $F \cap F' \subseteq F \cap G$ for all $F' \in \Delta$ with $F' \neq F$. A connected simplicial complex Δ is called a *tree*, if every nonempty connected subcomplex of Δ has a leaf. This notion generalize trees from graph theory. Note that $\Delta_{n,m}$ is a tree, in the sense of the above definition.

According to [9, Corollary 1.6], if I is the facet ideal associated to a tree (which is the case for $I_{n,m}$), it follows that S/I would be pretty clean. However, there is a mistake in the second line of the proof of [9, Proposition 1.4], and therefore, this result might be wrong in general. On the other hand, if $I \subset S$ is a pretty clean monomial ideal, it is known that $\operatorname{sdepth}(S/I) = \operatorname{depth}(S/I)$, see [12, Proposition 18] for further details.

1 Main results

We recall the well known Depth Lemma, see for instance [19, Lemma 1.3.9] or [18, Lemma 3.1.4].

Lemma 1.1. *(Depth Lemma) If $0 \to U \to M \to N \to 0$ is a short exact sequence of modules over a local ring S, or a Noetherian graded ring with S_0 local, then*

a) $\operatorname{depth} M \geq \min\{\operatorname{depth} N, \operatorname{depth} U\}$.

b) $\operatorname{depth} U \geq \min\{\operatorname{depth} M, \operatorname{depth} N + 1\}$.

c) $\operatorname{depth} N \geq \min\{\operatorname{depth} U - 1, \operatorname{depth} M\}$.

In [14], Asia Rauf proved the analog of Lemma 1.1(a) for sdepth:

Lemma 1.2. *Let $0 \to U \to M \to N \to 0$ be a short exact sequence of $\mathbb{Z}^n$-graded S-modules. Then:*

$$\operatorname{sdepth}(M) \geq \min\{\operatorname{sdepth}(U), \operatorname{sdepth}(N)\}.$$

Our main result is the following theorem.

Theorem 1.3. $\operatorname{sdepth}(S/I_{n,m}) = \operatorname{depth}(S/I_{n,m}) = n+1-\left\lfloor\frac{n+1}{m+1}\right\rfloor - \left\lceil\frac{n+1}{m+1}\right\rceil$.

Proof. We use induction on $m \geq 1$ and $n \geq m$. The case $m=1$ is trivial. The case $m=2$ follows from [13, Lemma 2.8] and [17, Lemma 4].

We assume $m \geq 3$. If $n=m$, then $\operatorname{sdepth}(S/I_{n,m}) = \operatorname{depth}(S/I_{n,m}) = m-1$, since $I_{n,n} = (x_1 \cdots x_n)$ is principal. Assume $m+1 \leq n \leq 2m-1$. Note that $I_{n,m} = x_m(I_{n,m} : x_m)$. We have $\operatorname{sdepth}(S/I_{n,m}) = \operatorname{sdepth}(S/(I_{n,m} : x_m))$, by [3, Theorem 1.4]. Also, we obviously have $\operatorname{depth}(S/I_{n,m}) = \operatorname{depth}(S/(I_{n,m} : x_m))$. On the other hand, $S/(I_{n,m} : x_m)$ is isomorphic to $S'/(I_{n-1,m-1})[y]$, where $S' = K[x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_n]$ and therefore, by induction hypothesis and [11, Lemma 3.6], we get $\operatorname{sdepth}(S/I_{n,m}) = \operatorname{depth}(S/I_{n,m}) = 1 + (n - \left\lfloor\frac{n}{m}\right\rfloor - \left\lceil\frac{n}{m}\right\rceil) = 1+n-3 = n-2$, as required.

It remains to consider the case $m \geq 3$ and $n \geq 2m$. Let $k := \left\lfloor\frac{n+1}{m+1}\right\rfloor$ and $a = n+1-k(m+1)$. We denote $\varphi(n,m) := n+1-\left\lfloor\frac{n+1}{m+1}\right\rfloor - \left\lceil\frac{n+1}{m+1}\right\rceil$. One can easily see that $\varphi(n,m) = \begin{cases} n+1-2k, & a=0 \\ n-2k, & a \neq 0 \end{cases}$.

We consider the ideals $L_0 := I_{n,m}$ and $L_j := (L_{j-1} : x_{j(m+1)-1})$, where $1 \leq j \leq k$. We denote $U_j := (L_{j-1}, x_{j(m+1)-1})$ for all $1 \leq j \leq k$. We have the following short exact sequences:

$$(\mathcal{S}_k):\ \ 0 \longrightarrow S/L_j \stackrel{\cdot x_{j(m+1)-1}}{\longrightarrow} S/L_{j-1} \longrightarrow S/U_j \longrightarrow 0,\ \ 1 \leq j \leq k.$$

We denote $u_i := x_i \cdots x_{i+m-1}$, for $1 \leq i \leq n-m+1$. Note that $G(L_0) = \{u_1, \ldots, u_{n-m+1}\}$, $G(L_1) = \{\frac{u_1}{x_m}, \ldots, \frac{u_m}{x_m}, u_{m+2}, \ldots, u_{n-m+1}\}$, because $u_{m+1} \in (u_m/x_m)$, and, also, $G(U_1) = \{x_m, u_{m+1}, \ldots, u_{n-m+1}\}$. Moreover, one can easily check that:

$$L_j = (\frac{u_1}{x_m}, \ldots, \frac{u_m}{x_m}, \frac{u_{m+2}}{x_{2m+1}}, \ldots, \frac{u_{2m+1}}{x_{2m+1}}, \ldots, \frac{u_{(m+1)j-m}}{x_{(m+1)j-1}}, \ldots, \frac{u_{(m+1)j-1}}{x_{(m+1)j-1}}, u_{(m+1)j+1}, \ldots, u_{n-m+1}),$$

for all $1 \leq j \leq k-1$. It follows that:

$$U_{j+1} = (\frac{u_1}{x_m}, \ldots, \frac{u_m}{x_m}, \ldots, \frac{u_{(m+1)j-m}}{x_{(m+1)j-1}}, \ldots, \frac{u_{(m+1)j-1}}{x_{(m+1)j-1}}, x_{(m+1)(j+1)-1}, u_{(m+1)(j+1)}, \ldots, u_{n-m+1}),$$

for all $1 \leq j \leq k-1$. Also, we have:

$$L_k = (\frac{u_1}{x_m}, \ldots, \frac{u_m}{x_m}, \ldots, \frac{u_{(m+1)(k-1)-m}}{x_{(m+1)(k-1)-1}}, \ldots, \frac{u_{(m+1)(k-1)-1}}{x_{(m+1)(k-1)-1}}, \frac{u_{(m+1)k-m}}{x_{(m+1)k-1}}, \ldots, \frac{u_t}{x_{(m+1)k-1}}),$$

where $t = n-m$ if $a=m$, or $t = n-m+1$ otherwise.

Note that $|G(L_k)| = m(k-1)+(t+1)-(m+1)k+m = t+1-k$ and, moreover, $L_k \cong I_{t+m-k-1,m-1}S$. Thus, by induction hypothesis and [11, Lemma 3.6], we have $\operatorname{depth}(S/L_k) = \operatorname{sdepth}(S/L_k) = n-(t+m-k-1)+\varphi(t+m-k-1,m-1) = n+1-\left\lfloor\frac{t+m-k}{m}\right\rfloor - \left\lceil\frac{t+m-k}{m}\right\rceil$.

If $a=m$, then $t=n-m$, $n=k(m+1)+m-1$, $t+m-k = n-k = (k+1)m-1$ and thus $\operatorname{depth}(S/L_k) = \operatorname{sdepth}(S/L_k) = n+1-k-(k+1) = n-2k = \varphi(n,m)$. If $a=0$, then $t+m-k = km$ and thus $\operatorname{depth}(S/L_k) = \operatorname{sdepth}(S/L_k) = n+1-2k$.

If $0 < a < m$, then $t+m-k = km+a$ and thus $\operatorname{depth}(S/L_k) = \operatorname{sdepth}(S/L_k) = n-2k$. In all the cases, we have $\operatorname{depth}(S/L_k) = \operatorname{sdepth}(S/L_k) = \varphi(n,m)$.

Note that $S/U_1 \cong K[x_{m+1},\ldots,x_n]/(u_{m+1},\ldots,u_{n-m+1})[x_1,\ldots,x_{m-1}]$ and therefore, by induction hypothesis, $\operatorname{depth}(S/U_1) = \operatorname{sdepth}(S/U_1) = m-1+\varphi(n-m,m) = n-\left\lfloor \frac{n-m+1}{m+1} \right\rfloor - \left\lceil \frac{n-m+1}{m+1} \right\rceil$. Note that $\frac{n-m+1}{m+1} = k-1+\frac{a+1}{m+1}$ and therefore $\left\lceil \frac{n-m+1}{m+1} \right\rceil = k$. On the other hand, if $a < m$ then $\left\lfloor \frac{n-m+1}{m+1} \right\rfloor = k-1$ and if $a = m$ then $\left\lfloor \frac{n-m+1}{m+1} \right\rfloor = k$. It follows that $\operatorname{depth}(S/U_1) = \operatorname{sdepth}(S/U_1) = \begin{cases} n+1-2k, & a<m \\ n-2k, & a=m \end{cases} \geq \varphi(n,m)$.

Moreover, $\operatorname{depth}(S/U_1) = \operatorname{sdepth}(S/U_1) = \varphi(n,m)$ if and only if $a = 0$ or $a = m$. Otherwise, $\operatorname{depth}(S/U_1) = \operatorname{sdepth}(S/U_1) = \varphi(n,m)+1$. Assume $a = 0$ or $a = m$. From the exact sequence $(\mathcal{S}_1) 0 \to S/L_1 \to S/L_0 \to S/U_1 \to 0$, Lemma 1.1 and Lemma 1.2, it follows that $\operatorname{sdepth}(S/L_0) \geq \operatorname{depth}(S/L_0) = \varphi(n,m)$. On the other hand, since $L_k = (L_0 : x_m x_{2m+1} \cdots x_{k(m+1)-1})$, for example by [5, Proposition 2.7], $\varphi(n,m) = \operatorname{sdepth}(S/L_k) \geq \operatorname{sdepth}(S/L_0) \geq \varphi(n,m)$. Thus, $\operatorname{sdepth}(S/L_k) = \varphi(n,m)$.

It remains to consider the case when $1 < a < m-1$. We claim that:

$$(*) \operatorname{sdepth}(S/U_j) \geq \operatorname{depth}(S/U_j) \geq \varphi(n,m) \textit{ for all } 2 \leq j \leq k.$$

Assume this is the case. Using 1.1, 1.2 and the short exact sequences $(\mathcal{S}_k)$, we get, inductively, that $\operatorname{sdepth}(S/L_j) \geq \operatorname{depth}(S/L_j) = \varphi(n,m)$ for all $j < k-1$. Again, using for example [5, Proposition 2.7], we get $\operatorname{sdepth}(S/L_0) = \varphi(n,m)$.

In order to complete the proof, we need to show $(*)$. Note that $U_k = (V_k, x_{(m+1)k-1})$, where $V_k = (\frac{u_1}{x_m},\ldots,\frac{u_m}{x_m},\ldots,\frac{u_{(m+1)j-m}}{x_{(m+1)j-1}},\ldots,\frac{u_{(m+1)(k-1)-1}}{x_{(m+1)(k-1)-1}}) \cong I_{mk-2,m-1}S$. By induction hypothesis and [11, Lemma 3.6], it follows that $\operatorname{sdepth}(S/U_k) = \operatorname{depth}(S/U_k) = n-(mk-2)-1+\varphi(mk-2,m-1) = n-\left\lfloor \frac{mk-1}{m} \right\rfloor - \left\lceil \frac{mk-1}{m} \right\rceil = n-(k-1)-k = n-2k+1 = \varphi(n,m)+1$.

If $1 \leq j < k$, we have $S/U_j \cong (S/V_j \otimes_S S/W_jS)/(x_{(m+1)j-1})(S/V_j \otimes_S S/W_jS)$, where $V_j = (\frac{u_1}{x_m},\ldots,\frac{u_m}{x_m},\ldots,\frac{u_{(m+1)j-m}}{x_{(m+1)j-1}},\ldots,\frac{u_{(m+1)j-1}}{x_{(m+1)j-1}})$ and $W_j = (u_{(m+1)(j+1)},\ldots,u_{n-m+1})$. Since $x_{(m+1)j-1}$ is regular on $S/V_j \otimes_S S/W_j$ by [14, Corollary 1.12] and [14, Theorem 3.1] or [5, Theorem 1.2], it follows that $\operatorname{depth}(S/U_j) = \operatorname{depth}(S/V_j \otimes_S S/W_j) - 1 = \operatorname{depth}(S/V_j) + \operatorname{depth}(S/W_j) - n - 1$ and $\operatorname{sdepth}(S/U_j) = \operatorname{sdepth}(S/V_j \otimes_S S/W_j) - 1 \geq \operatorname{sdepth}(S/V_j) + \operatorname{sdepth}(S/W_j) - n - 1$.

On the other hand, $V_j \cong I_{m(j+1)-2,m-1}S$ and thus, by induction hypothesis, $\operatorname{sdepth}(S/V_j) = \operatorname{depth}(S/V_j) = n+1-\left\lfloor \frac{m(j+1)-1}{m} \right\rfloor - \left\lceil \frac{m(j+1)-1}{m} \right\rceil = n-2j$. Also, $W_j \cong I_{n-(m+1)(j+1)+1,m}$ and, by induction hypothesis, we have $\operatorname{sdepth}(S/W_j) = \operatorname{depth}(S/W_j) = n+1-\left\lfloor \frac{n-(m+1)(j+1)+2}{m+1} \right\rfloor - \left\lceil \frac{n-(m+1)(j+1)+2}{m+1} \right\rceil = n+1+2(j+1)-\left\lfloor \frac{n+2}{m+1} \right\rfloor - \left\lceil \frac{n+2}{m+1} \right\rceil$.

It follows that $\operatorname{sdepth}(S/U_j) = \operatorname{depth}(S/U_j) = n+2-\left\lfloor \frac{n+2}{m+1} \right\rfloor - \left\lceil \frac{n+2}{m+1} \right\rceil \geq \varphi(n,m)$, since either $\left\lfloor \frac{n+2}{m+1} \right\rfloor = \left\lfloor \frac{n+1}{m+1} \right\rfloor$ and $\left\lceil \frac{n+2}{m+1} \right\rceil = \left\lceil \frac{n+1}{m+1} \right\rceil$, either $\left\lfloor \frac{n+2}{m+1} \right\rfloor = \left\lfloor \frac{n+1}{m+1} \right\rfloor + 1$ and $\left\lceil \frac{n+2}{m+1} \right\rceil = \left\lceil \frac{n+1}{m+1} \right\rceil$ or either $\left\lfloor \frac{n+2}{m+1} \right\rfloor = \left\lfloor \frac{n+1}{m+1} \right\rfloor$ and $\left\lceil \frac{n+2}{m+1} \right\rceil = \left\lceil \frac{n+1}{m+1} \right\rceil + 1$. □

Example 1.4. Let $I_{6,3} = (x_1x_2x_3, x_2x_3x_4, x_3x_4x_5, x_4x_5x_6) \subset S := K[x_1,\ldots,x_6]$. Note that $\varphi(7,4) = 7 - \lfloor \frac{7}{4} \rfloor - \lceil \frac{7}{4} \rceil = 4$. Let $L_0 = I_{6,3}$, $L_1 = (L_0 : x_3) = (x_1x_2, x_2x_4, x_4x_5)$ and $U_1 = (L_0, x_3) = (x_3, x_4x_5x_6)$. Since $L_1 \cong I_{4,2}S$, it follows that $\operatorname{depth}(S/L_1) = \operatorname{sdepth}(S/L_1) = \operatorname{depth}(S/I_{4,2}S) = 2 + \operatorname{depth}(K[x_1,\ldots,x_4]/I_{4,2}) = 2 + \varphi(4,2) = 4$.

On the other hand, since U_1 is a complete intersection, $\operatorname{depth}(S/U_1) = \operatorname{sdepth}(S/U_1) = 4$. We consider the short exact sequence $0 \to S/L_1 \to S/L_0 \to S/U_1 \to 0$. By Lemma 1.2, it follows that $\operatorname{sdepth}(S/L_0) \geq 4$. On the other hand, since $L_1 = (L_0 : x_3)$, one has $\operatorname{sdepth}(S/L_0) \leq \operatorname{sdepth}(S/L_1) = 4$. Thus $\operatorname{sdepth}(S/L_0) = 4$. Also, by Lemma 1.1, $\operatorname{depth}(S/L_0) = 4$.

In the following, we present another way to prove that $\operatorname{sdepth}(S/I_{n,m}) \leq \varphi(n,m)$.

Let $\mathcal{P} \subset 2^{[n]}$ be a poset. If $C, D \subset [n]$, the *interval* $[C,D]$ consist in all the subsets X of $[n]$ such that $C \subset X \subset D$. Let $\mathbf{P} : \mathcal{P} = \bigcup_{i=1}^{r} [F_i, G_i]$ be a partition of $\mathcal{P}$, i.e. $[F_i, G_i] \cap [F_j, G_j] = \emptyset$ for all $i \neq j$. We denote $\operatorname{sdepth}(\mathbf{P}) := \min_{i\in[r]} |D_i|$. Also, we define the Stanley depth of $\mathcal{P}$, to be the number

$$\operatorname{sdepth}(\mathcal{P}) = \max\{\operatorname{sdepth}(\mathbf{P}) :\ \mathbf{P} \textit{ is a partition of} \mathcal{P}\}.$$

Now, for $d \in \mathbb{N}$ and $\sigma \in \mathcal{P}$, we denote

$$\mathcal{P}_d = \{\tau \in \mathcal{P} \ :\ |\tau| = d\}\ ,\ \mathcal{P}_{d,\sigma} = \{\tau \in \mathcal{P}_d \ :\ \sigma \subset \tau\}.$$

Note that if $\sigma \in \mathcal{P}$ such that $P_{d,\sigma} = \emptyset$, then $\operatorname{sdepth}(\mathcal{P}) < d$. Indeed, let $\mathbf{P} : \mathcal{P} = \bigcup_{i=1}^{r} [F_i, G_i]$ be a partition of $\mathcal{P}$ with $\operatorname{sdepth}(\mathcal{P}) = \operatorname{sdepth}(\mathbf{P})$. Since $\sigma \in \mathcal{P}$, it follows that $\sigma \in [F_i, G_i]$ for some i. If $|G_i| \geq d$, then it follows that $\mathcal{P}_{d,\sigma} \neq \emptyset$, since there are subsets in the interval $[F_i, G_i]$ of cardinality d which contain σ, a contradiction. Thus, $|G_i| < d$ and therefore $\operatorname{sdepth}(\mathcal{P}) < d$.

We recall the method of Herzog, Vladoiu and Zheng [11] for computing the Stanley depth of S/I and I, where I is a squarefree monomial ideal. Let $G(I) = \{u_1,\ldots,u_s\}$ be the set of minimal monomial generators of I. We define the following two posets:

$$\mathcal{P}_I := \{\sigma \subset [n] :\ u_i | x_\sigma := \prod_{j\in\sigma} x_j \textit{ for some } i\ \} \textit{ and } \mathcal{P}_{S/I} := 2^{[n]} \setminus \mathcal{P}_I.$$

Herzog Vladoiu and Zheng proved in [11] that $\operatorname{sdepth}(I) = \operatorname{sdepth}(\mathcal{P}_I)$ and $\operatorname{sdepth}(S/I) = \operatorname{sdepth}(\mathcal{P}_{S/I})$.

The above method is useful to give upper bounds for the $\operatorname{sdepth}(S/I)$, where $I \subset S$ is a monomial ideal, and, in particular cases, to compute the exact value of $\operatorname{sdepth}(S/I)$. That's exactly the case for $S/I_{n,m}$!

Let $\mathcal{P} := \mathcal{P}_{S/I_{n,m}}$. We denote $k = \lfloor \frac{n}{m+1} \rfloor$ and we define

$$\sigma = \bigcup_{j=0}^{k-1} \{1 + j(m+1), 2 + j(m+1), \ldots, m - 1 + j(m+1)\}.$$

We consider two cases.

(a) If $n=(k+1)(m+1)-1$ or $n=(k+1)(m+1)-2$, let $\tau=\sigma\cup\{k(m+1)+1, k(m+1)+2,\ldots,k(m+1)+m-1\}$. Note that $|\tau|=(k+1)(m-1)$ and $\mathcal{P}_{d,\tau}=\emptyset$, for $d=|\tau|+1$. Indeed, $u=\prod_{j\in\tau}x_j\notin I_{n,m}$, but $x_iu\in I_{n,m}$ for all $i\notin\tau$.

(b) If n is not as in the case (a), let $\tau=\sigma\cup\{k(m+1),\ldots,n\}$. Note that $n-|\tau|=2k-1$ and $\mathcal{P}_{d,\tau}=\emptyset$, for $d=|\tau|+1$. Indeed, $u=\prod_{j\in\tau}x_j\notin I_{n,m}$, but $x_iu\in I_{n,m}$ for all $i\notin\tau$.

Therefore $\operatorname{sdepth}(S/I_{n,m})\leq|\tau|$, in both cases. On the other hand, one can easily check that $|\tau|=n+1-\lfloor\frac{n+1}{m+1}\rfloor-\lceil\frac{n+1}{m+1}\rceil$. Therefore $\operatorname{sdepth}(S/I_{n,m})\leq\varphi(n,m)$.

Remark 1.5. One possible way to generalize Theorem 1.3 and [17, Theorem 6], in the same time, would be to prove that $\operatorname{sdepth}(S/I_{n,m}^k)=\operatorname{depth}(S/I_{n,m}^k)$ for any $k\geq 1$. Furthermore, we might conjecture that if Δ is a simplicial tree, then $\operatorname{sdepth}(S/I(\Delta)^k)=\operatorname{depth}(S/I(\Delta)^k)$ for any $k\geq 1$.

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